

Rotational Motion 2

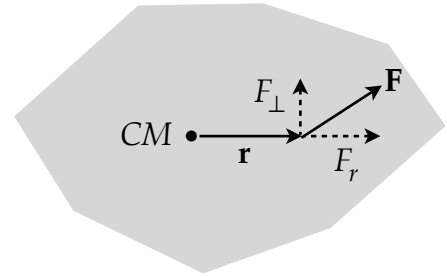
Right handers, go over there, left handers over here. The rest of you, come with me.
— Yogi Berra

Torque

Motion of a rigid body, like motion of any system of particles, is changed by the effects of external forces. A rigid body's motion consists of motion of the CM plus rotation about the CM; external forces can change one or both of these parts of the motion.

Shown is a rigid body, with an external force \mathbf{F} applied to it at a point specified by the position vector \mathbf{r} , relative to the CM as origin.

At first we take the force vector \mathbf{F} to lie in the plane of the drawing. This vector can be broken up into a component along \mathbf{r} (F_r) and a component perpendicular to \mathbf{r} (F_\perp).



Acting alone, the radial component F_r would make the CM accelerate to the right along the line of \mathbf{r} , but it would not produce any rotation *about* the CM. Now suppose there is a fixed axle passing through the body, perpendicular to the page and passing through the CM. Then normal forces exerted on the body by the axle would cancel the effect of F_r and the body would not move at all. We will ignore this component in what follows. On the other hand, F_\perp would make the body rotate about the axle. It is this *rotational* effect that we are interested in here.

As the body turns counter-clockwise through angle $d\theta$ during time dt , the point at which F_\perp is applied moves through distance $ds = r d\theta$, so the work done by it is $rF_\perp d\theta$. This infinitesimal work causes an infinitesimal increase in the *rotational* kinetic energy:

$$rF_\perp d\theta = d\left(\frac{1}{2}I\omega^2\right).$$

The quantity rF_\perp , acting as the body *rotates* through angle $d\theta$, produces an infinitesimal change in the *rotational* kinetic energy. This suggests that the quantity rF_\perp is a useful

measure of the effectiveness of the force in producing *angular* motion, i.e., *rotation*. We call this quantity the **torque** of the force, denoted by the Greek letter τ :

$$\tau = rF_{\perp}.$$

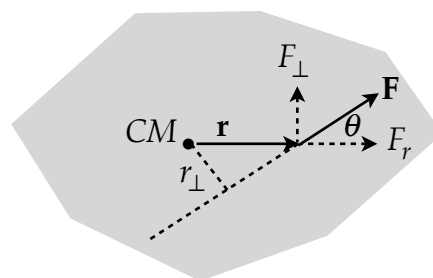
Actually, torque is a vector quantity. We have introduced the relevant *component* of the torque, in the case of a rigid body constrained to rotate about a fixed axle, and with applied force perpendicular to that axle. (In such a case the torque has only one component.) A more general definition will be given later.

Moment arm and torque

Since torque involves the distance r , it depends on the location of the reference point about which the body rotates. For the example considered here, it is the point where the axle passes through the plane of the diagram. (We have taken that point to be the CM for simplicity, but that is only one choice.)

To specify a torque requires specifying the reference point about which it acts.

Shown is the same situation discussed above. We see that since $F_{\perp} = F \sin \theta$ the torque's magnitude is given by $\tau = rF \sin \theta$. But this can also be written as $\tau = r_{\perp}F$, where $r_{\perp} = r \sin \theta$. The distance r_{\perp} is called the **moment arm** of the force.



Moment arm of a force	The moment arm of a force in producing a torque is the perpendicular distance from the reference point to the line along which the force acts.
-----------------------	--

In terms of this definition:

The magnitude of the torque is the product of the moment arm and the magnitude of the force.

If a force acts directly toward or away from the reference point, its moment arm is zero and the force produces no torque about that reference point. The moment arm is a maximum when $F \perp r$, in which case the moment arm is simply r .

In the case discussed so far, torques produce either clockwise or counter-clockwise rotation about the axle. One can choose which of these directions to count as positive, and a torque that will produce positive rotation is counted as a positive torque.

The three-dimensional aspects of the direction of the torque vector will be discussed later.

Relation between torque and angular acceleration

It is relatively easy to compute the torques, about a given point, due to external forces on a rigid body. It is much more subtle to relate these torques to the actual motion of the body, if it is not constrained by a fixed axle. Many of these subtleties are beyond the scope of this course. Even if the body is constrained to rotate about a fixed axle, there can be some subtle aspects, as will be discussed below.

Only if the body is *symmetric* about its rotation axis is the situation easy to analyze. Here “symmetric” means that each point mass in the body has an identical “mirror image” point mass with the same mass and at the same distance directly on the other side of the axis. For such a body the symmetry axis automatically passes through the CM. Bodies with uniform mass density which are figures of revolution about the axis (wheels, pulleys, etc.) have such symmetry.

For symmetric bodies, as will be shown later, there is a simple and useful relation between the net torque due to external forces and the angular acceleration of the body about the fixed axis:

Torque and angular acceleration (symmetric body, fixed axis)	$\tau_{tot} = I\alpha$
---	------------------------

This relation is one case of what is sometimes called the “rotational 2nd law”.

Rolling

An important example of a symmetric body rotating about its symmetry axis is the rolling motion of a uniform figure of revolution, such as a wheel or a ball.

Rolling is motion in which the point of contact between the moving object and the surface over which it is moving is always instantaneously at rest.

That is, the moving body does not *slide* on the surface.

To see the consequences of this requirement, it is useful to consider the motion in the frame moving with the CM of the body. In that frame, the motion of the rolling body is nothing but rotation about its symmetry axis, while the surface passes *backwards* underneath it with speed v_{CM} . A point on the outer rim of the body will move around the rim with speed $v = R\omega$, where R is the radius. If the point of contact is always to be at rest relative to the surface, the backward speed of the lowest point on the rim must equal the speed of the backward moving surface. This gives the **rolling condition**:

Rolling condition	$v_{CM} = R\omega$
-------------------	--------------------

If the body accelerates while still not slipping, then this condition must hold at all times, and we must have $a_{CM} = R\alpha$. An example is a wheel rolling down an incline.

For a rolling symmetric body the kinetic energy is

Kinetic energy of rolling	$K = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I\omega^2$
---------------------------	---

This formula has many applications.